Transient Electromagnetic Soundings
"Abridged Notes"

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Time Domain (Transient) Electromagnetic (TEM) Method

Fundamental Electromagnetic Quantities and Units

(Système Internationale, SI, or MKS)

Symbol	Quantity	Units
q	electric charge	coulomb, C
ρ_f	electric charge density, free	C/m^2
I	electric current	ampere, A = C/s
$\overrightarrow{J}$	electric or conduction current density	A/m^2
U	electric potential	volt, V
V	electric potential difference or voltage	volt, V
$\overrightarrow{D}$	electric displacement or electric flux density	C/m^2
$\overrightarrow{E}$	electric field strength	V/m
Φ_m	magnetic flux	Weber, Wb = V-s
$\overrightarrow{B}$	magnetic induction or magnetic flux density	tesla, T = weber/m^2, Wb/m^2
$\overrightarrow{H}$	magnetic field strength	A/m
E	energy	joule, J = N-m = V-C
P	power	Watt, W = J/s
σ	electric conductivity	siemen/m, S/m, or mho/m
R	resistance	ohm, Ω
L	inductance	henry, H
C	capacitance	farad, F
ρ	electric resistivity	ohm-m, Ω-m
ϵ	electrical permittivity or dielectric constant	farad/m, F/m
μ	magnetic permeability	henry/m, H/m

Maxwell's macroscopic equations

$$\begin{aligned}
\nabla \cdot \overrightarrow{D} &= \rho_f && \text{(Gauss' Law)} \\
\nabla \cdot \overrightarrow{B} &= 0 && \text{(Gauss' Law)} \\
\nabla \times \overrightarrow{E} &= -\frac{\partial}{\partial t}\overrightarrow{B} && \text{(Faraday's Law)} \\
\nabla \times \overrightarrow{H} &= \overrightarrow{J} + \frac{\partial}{\partial t}\overrightarrow{D} && \text{(Ampere-Maxwell Law)}
\end{aligned}$$

Some Interrelations

$$\begin{aligned}
\overrightarrow{D} &= \epsilon \overrightarrow{E} \\
\overrightarrow{B} &= \mu \overrightarrow{H} \\
\overrightarrow{J} &= \sigma \overrightarrow{E}
\end{aligned}$$

$$
\begin{aligned}
V &= IR \\
L &= \Phi_m / I \\
C &= q/V \\
\sigma &= 1/\rho
\end{aligned}
$$

Transient Electromagnetic (TEM) Applications

1. Prospecting for conductors
   a. Massive sulfide ore deposits
   b. Hot water geothermal reservoirs
2. Subsurface sounding/mapping
   a. Petroleum, coal, and oil shale exploration
   b. Weathered zone in diamond prospecting
   c. Groundwater exploration
   d. Contaminant mapping
   e. Permafrost mapping

WHAT IS TRANSIENT EM (TEM)?

The principle of the Transient E.M. (TEM) method of geophysical prospecting is very simply, that current flowing in a transmitter loop sets up a magnetic field which when switched off induces eddy currents to flow in any good electrical conductor in the ground. These eddy currents set up a secondary magnetic field which can be detected by a receiver loop as a time-dependant decaying voltage.

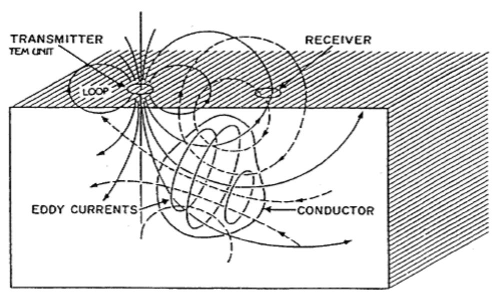


FIG. 1 PRIMARY MAGNETIC FIELD
SECONDARY MAGNETIC FIELD

HOW IS TEM USED?

The recording of the 'transients' is a means of detecting conductors in the ground. The decaying transient can be described by a number of measurement channels recording the voltage at various delay times (see figure 2) during the "quiet time" between current pulses.
The character of this decay (duration, amplitude, etc.) depends on the conductivity, shape and size, and depth and attitude of the conductor and its position with respect to the receiver loop and can be used to provide information on all this factors.

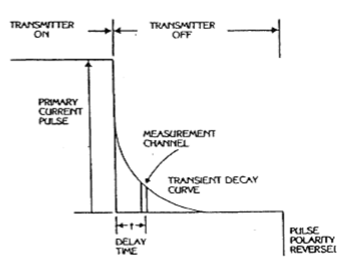
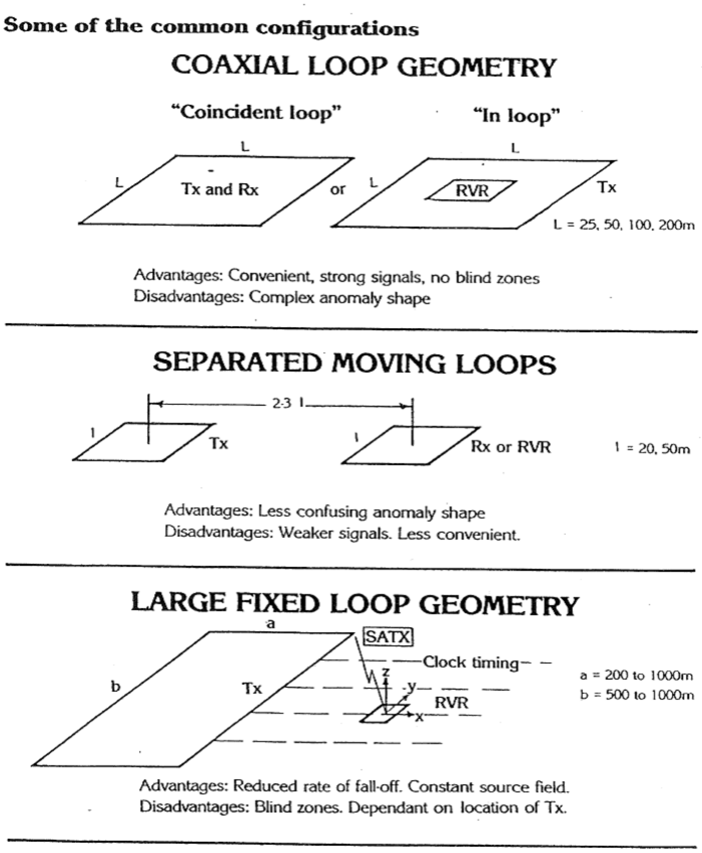


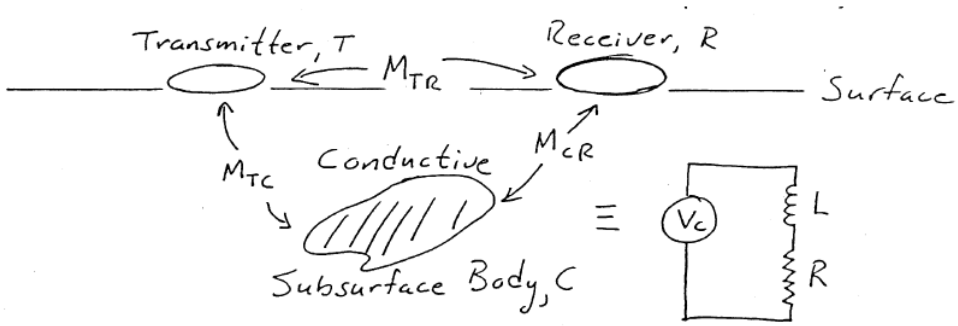
FIG. 2 Schematic diagram of TEM operation

A particular advantage of transient E.M. systems over continuous waves systems is the fact that the measurements are taken when the transmitted fields are switched off. This means that the sensitivity of the receiver can be a maximum to record the transient voltages only without having to cope with the much greater signal strength of the transmission field. It also means that a greater variety of loop configurations can be used including having the receiver loop in the same place as the transmitter loop for maximum signal reception.



TEM Circuit Analogy

(after Grant and West, 1965, p. 541-544)

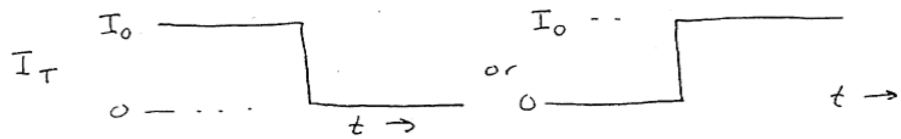


M_{TC}	:	mutual inductance (coupling) between T and C
M_{TR}	:	mutual inductance (coupling) between T and R
M_{CR}	:	mutual inductance (coupling) between C and R
V_C	:	voltage induced in C due to Faraday's law
L	:	equivalent inductance of C
R	:	equivalent resistance of C

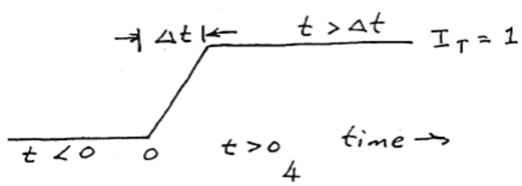
Applying Kirchkoff's second law (loop law) - sum of the voltages around a closed loop is zero

$$V_C - V_L - V_R = 0.$$

In theory we assume transmitter current, I_T, is a step function, i.e.,



but in actuality it is



where Δt is a small rise (or decay) time.

Voltage, V_R, measured at the receiver due to a time varying magnetic field caused by the current, I_C, in the conductor is (Appendix A):

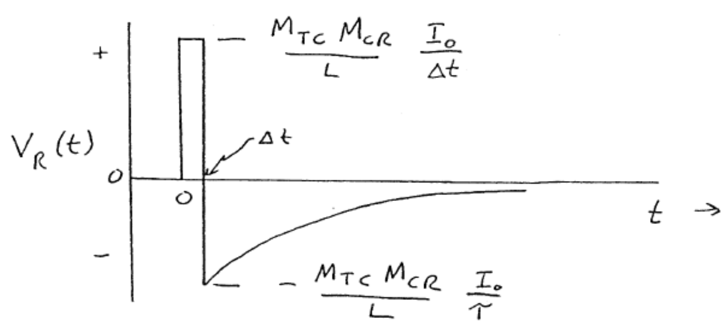
$$
\begin{aligned}
V_R(t) &= -M_{CR}\frac{d}{dt}I_C(t) \\
&= +\frac{M_{TC}M_{CR}}{L}I_0\left[\delta(t) - \frac{R}{L}e^{-\frac{R}{L}t}\right]
\end{aligned}
$$

where $\delta(t)$ is the impulse or delta function

$$
\begin{aligned}
\delta(t) &= 0,\ t < 0 \\
&= 1/\Delta t,\ 0 \leq t \leq \Delta t \\
&= 0,\ t > \Delta t
\end{aligned}
$$

and $\tau = L/R$ where τ is a function of σ, size, shape, and depth of C.

Graphically



The exponential term in $V_R(t)$ is proportional to the time rate of change in the secondary magnetic field (Faraday's law) and is

$$V_R(t) \propto \frac{1}{\tau}e^{-t/\tau}.$$

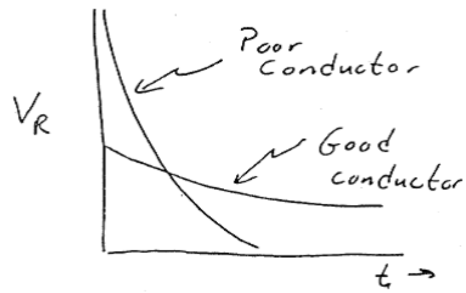
Key Characteristics of TEM Decay

Poor conductors, $(\tau = L/R)$ small

Large initial voltage, rapid V_R decay

Good conductors, $(\tau = L/R)$ large

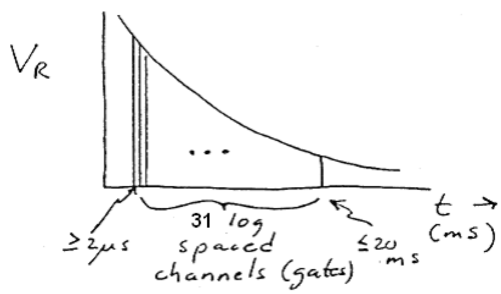
Smaller initial voltage, slower V_R decay

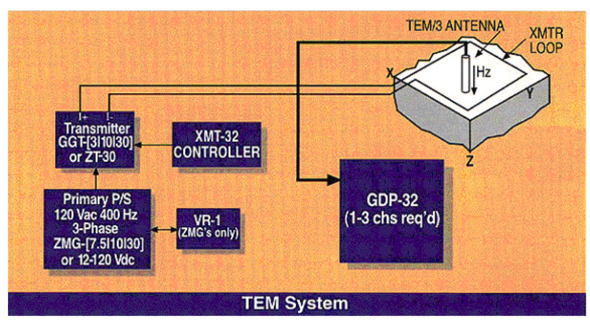


Typical τ for ore bodies is $100\mu\text{s} \lesssim \tau \lesssim 20\,\text{ms}$.

TEM Measurement

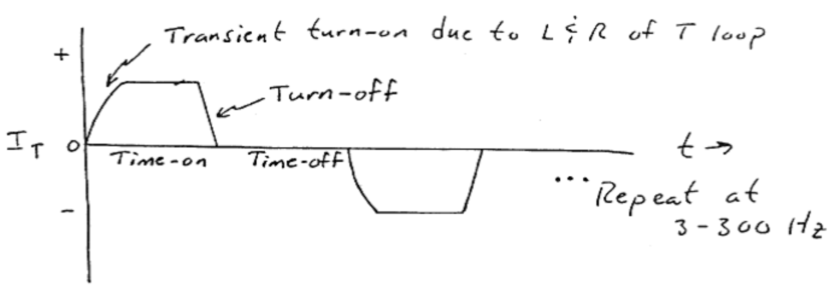
(Zonge GDP-32 Receiver)



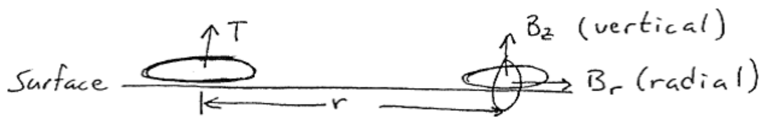


Time rate of decay of V_R can be measured for 3-axes of secondary magnetic field.

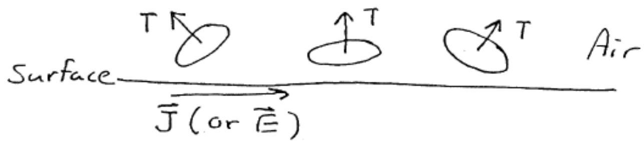
TEM Transmitter



Response of Homogeneous Conducting Half-Space



All transmitter loop configurations induce only horizontal currents in a homogeneous half-space (or one-dimensional earth).



Due to continuity in normal $\overrightarrow{J}$ with $\overrightarrow{J} = 0$ in air.

At $t = 0$ (transmitter turn-off)

Induced current in the earth is primarily at the surface and tries to maintain a magnetic field everywhere as it was just before turn-off (Lenz's law). This opposition to the change in magnetic field would best be accomplished by a current loop in the earth that would be an exact image of the transmitter current just before turn-off.

* No vertical component of current.

At $t = 0^+$

Current distribution decays and diffuses downward and outward. Current always flows horizontally with no downward component of current. This is described (e.g., Nabighian and Macnae, 1991) as the "smoke ring" concept or a system of equivalent current filaments.

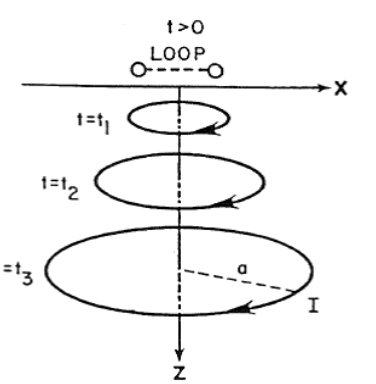


Fig. 11. System of equivalent current filaments, at various times after current interruption in the transmitter loop, showing their downward and outward movement (after Nabighian, 1979).

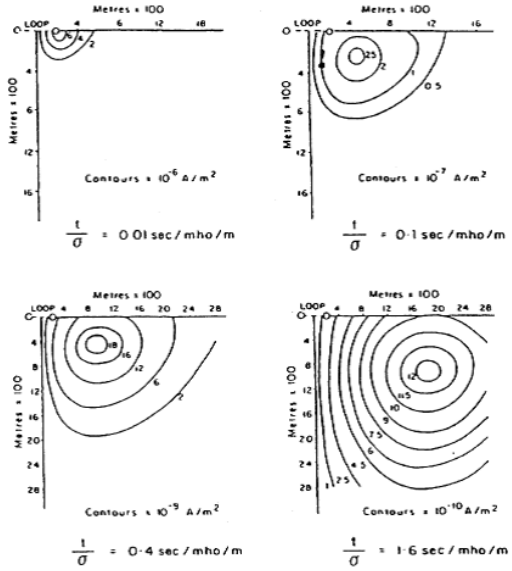


FIGURE 14. Computed contours of current density passing through loop centre (loop has dimensions 400 × 800 m (5)).

(Nabighian and Macnae, 1991)

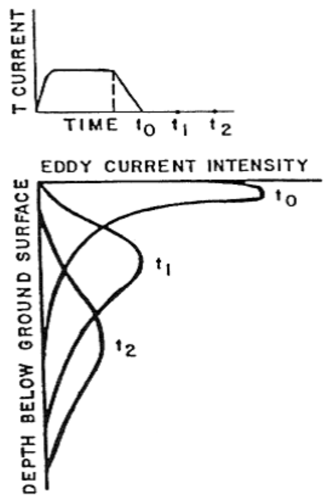
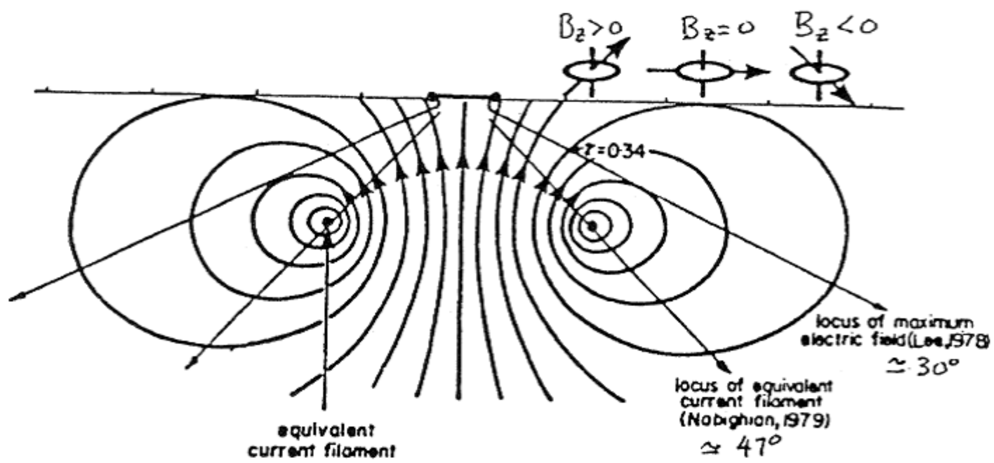


FIGURE 3. Schematic illustration of maximum current distribution intensity in a vertical plane.

(Geo-physi-con, undated)

Surface Magnetic Field and Current Density

H_S (and H_z) above a current filament at a constant time.



In cylindrical coordinates

(Nabighian and Macnae, 1991)

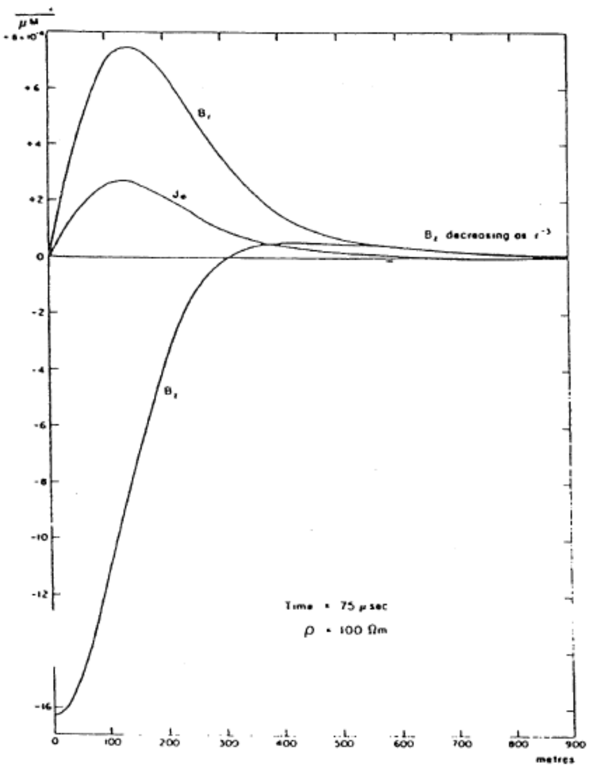


FIGURE 17. Magnetic field components at surface of homogeneous half-space.

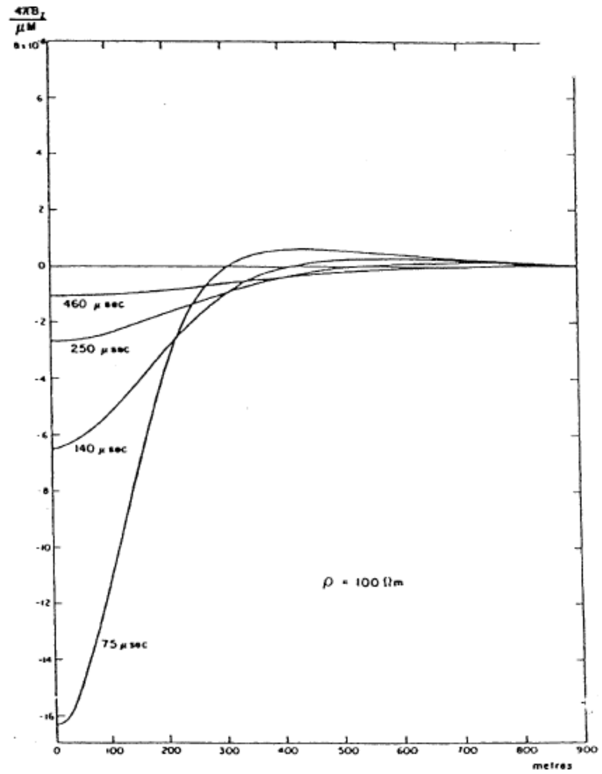


FIGURE 18. Vertical magnetic field component at surface of homogeneous half-space.

Diffusion Distance

The vertical location of the surface current maximum is (Nabighian and Macnae, 1991)

$$
\begin{aligned}
\delta &= \text{diffusion length or diffusion depth} \\
&= \left(\frac{2t}{\mu\sigma}\right)^{1/2} \\
&\propto t^{1/2}
\end{aligned}
$$

so the apparent vertical velocity of diffusion varied as $t^{-1/2}$. Often the *diffusion distance* is defined as

$$d = 2\pi\delta.$$

The peak of current (McNeill, 1980) is located at

$$r_{\max} \simeq d/5.2.$$

Time Stages in the Diffusion Process

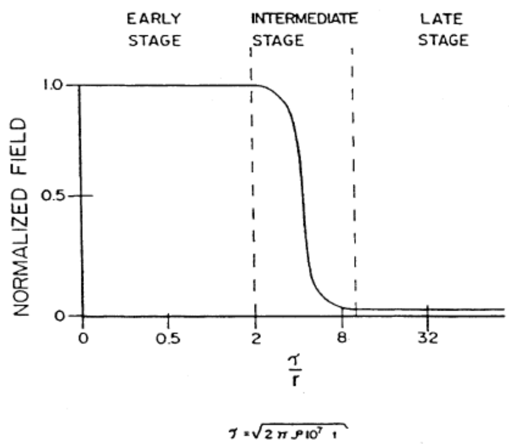


FIGURE 4. Behavior of time derivative of vertical magnetic field as a function of time. (Geo-physi-con, undated)

Using r as a characteristic dimension of the geometry, such as the distance between the centers of the T and R loops or the transmitter loop radius for a central "coaxial" loop array.

Early stage (time)

$$d \ll r$$

Intermediate stage (time)

$$d \approx r$$

Late stage (time)

$$d \gg r$$

$$(d/r \gtrsim 10)$$

Table 1. Early and late time asymptotic forms for a vertical magnetic dipole.

Component	Early time $t \to 0$	Late time $t \to \infty$
$h_r(t)$	$\dfrac{3mt^{1/2}}{\pi^{3/2}\sigma^{1/2}\mu^{1/2}r^4}$	$\dfrac{mr\mu^2\sigma^2}{128\pi t^2}$
$h_z(t)$	$\dfrac{m}{4\pi r^3}\left[1 - \dfrac{18t}{\sigma\mu r^2}\right]$	$-\dfrac{m\sigma^{3/2}\mu^{3/2}}{30\pi^{3/2}t^{3/2}}$
$e_\phi(t)$	$\dfrac{-3m}{2\pi\sigma r^4}$	$\dfrac{m\sigma^{3/2}\mu^{5/2}r}{40\pi^{3/2}t^{5/2}}$

Nabighian and Macnae (1991)

Where m is the magnetic moment equal to the product of current, loop area, and number of turns in the transmitter.

Measurement in TEM is of a voltage at the receiver. This voltage (via Faraday's law) is proportional to the time derivative of the magnetic field. The following page from Spies and Frischknecht (1991) tabulates this quantity for early and late times over a homogeneous half-space.

Electromagnetic Sounding

Table 2. Quasistatic fields of various sources on the surface of a homogeneous half-space in the time domain.

General Expression	D.C. Value $(t < 0)$	Early-Time Asymptote $(\theta r \gg 1, t \to 0)$		Late-Time Asymptote $(\theta r \ll 1, t \to \infty)$	
		Vertical Magnetic Dipole Source			
$e_\phi = \frac{-m}{2\pi\sigma r^4}\left[3\,\mathrm{erf}(\theta r) - \frac{2}{\sqrt{\pi}}\theta r(3 + 2\theta^2 r^2)e^{-\theta^2 r^2}\right]$ (loop-wire)	$e_\phi^0 = 0$	$e_\phi^e = \frac{-3m}{2\pi\sigma r^4}$	$\rho_a^e = \frac{-2\pi r^4}{3m} e_\phi$	$e_\phi^\ell = \frac{-m\sigma^{3/2}\mu_0^{5/2} r}{40\pi^{3/2} t^{5/2}}$	$\rho_a^\ell = \frac{m^{2/3}\mu_0^{5/3} r^{2/3}}{40^{2/3}\pi t^{5/3}}(-e_\phi)^{-2/3}$
$h_r = \frac{-m\theta^2}{2\pi r} e^{\frac{-\theta^2 r^2}{2}}\left[\left(1 + \frac{4}{\theta^2 r^2}\right) I_1\left(\frac{\theta^2 r^2}{2}\right) - I_0\left(\frac{\theta^2 r^2}{2}\right)\right]$ (perpendicular)	$h_r^0 = 0$	$h_r^e = \frac{-3m t^{1/2}}{\pi^{3/2}\sigma^{1/2}\mu_0^{1/2} r^4}$	$\rho_a^e = \frac{\pi^3 \mu_0 r^8}{9m^2 t} h_r^2$	$h_r^\ell = \frac{-m\sigma^2\mu_0^2 r}{128 t^2}$	$\rho_a^\ell = \frac{m^{1/2}\mu_0 r^{1/2}}{8(2\pi)^{1/2} t}(-h_r)^{-1/2}$
$\frac{\partial h_r}{\partial t} = \frac{m\theta^2}{2\pi r t} e^{\frac{-\theta^2 r^2}{2}}\left[(1 + \theta^2 r^2) I_0\left(\frac{\theta^2 r^2}{2}\right) - \left(2 + \theta^2 r^2 + \frac{4}{\theta^2 r^2}\right) I_1\left(\frac{\theta^2 r^2}{2}\right)\right]$		$\frac{\partial h_r^e}{\partial t} = \frac{-3m}{2\pi^{3/2}\sigma^{1/2}\mu_0^{1/2} r^4 t^{1/2}}$	$\rho_a^e = \frac{4\pi^3\mu_0 r^8 t}{9m^2}\left(\frac{\partial h_r}{\partial t}\right)^2$	$\frac{\partial h_r^\ell}{\partial t} = \frac{m\mu_0^2\sigma^2 r}{64 t^3}$	$\rho_a^\ell = \frac{m^{1/2}\mu_0 r^{1/2}}{8\pi^{1/2} t^{3/2}}\left(\frac{\partial h_r}{\partial t}\right)^{-1/2}$
$h_z = \frac{m}{4\pi r^3}\left[\frac{9}{2\theta^2 r^2}\mathrm{erf}(\theta r) - \mathrm{erf}(\theta r) - \frac{1}{\sqrt{\pi}}\left(\frac{9}{\theta r} + 4\theta r\right)e^{-\theta^2 r^2}\right]$ (horizontal coplanar)	$h_z^0 = \frac{-m}{4\pi r^3}$	$h_z^e = \frac{-m}{4\pi r^3}\left(1 - \frac{18t}{\sigma\mu_0 r^2}\right)$	$\rho_a^e = \frac{2\pi\mu_0 r^5}{9mt}\left(\frac{m}{4\pi r^3} + h_z\right)$	$h_z^\ell = \frac{m\sigma^{3/2}\mu_0^{3/2}}{30\pi^{3/2} t^{3/2}}$	$\rho_a^\ell = \frac{m^{2/3}\mu_0}{30^{2/3}\pi t}(h_z)^{-2/3}$
$\frac{\partial h_z}{\partial t} = \frac{m}{2\pi\sigma\mu_0 r^5}\left[9\,\mathrm{erf}(\theta r) - \frac{2\theta r}{\sqrt{\pi}}(9 + 6\theta^2 r^2 + 4\theta^4 r^4)e^{-\theta^2 r^2}\right]$		$\frac{\partial h_z^e}{\partial t} = \frac{9m}{2\pi\sigma\mu_0 r^5}$	$\rho_a^e = \frac{2\pi\mu_0 r^5}{9mt}\frac{\partial h_z}{\partial t}$	$\frac{\partial h_z^\ell}{\partial t} = \frac{-m\sigma^{3/2}\mu_0^{5/2}}{20\pi^{3/2} t^{5/2}}$	$\rho_a^\ell = \frac{m^{2/3}\mu_0}{20^{2/3}\pi t^{5/3}}\left(\frac{-\partial h_z}{\partial t}\right)^{-2/3}$

$\theta = \left(\frac{\sigma\mu_0}{4t}\right)^{1/2}$

(Spies and Frischknecht, 1991)

Characteristics of Early and Late Time TEM Vertical Magnetic Field Measurements

Magnetic Field Measurements
- Early time, $V \propto \rho/r^5$
  - Diffusion distance is much less than T-R separation
  - Diffusion current concentrated near T loop
  - Voltage is independent of t
  - Voltage is linearly proportional to ρ
  - Voltage falls off sharply with distance, $\propto 1/r^5$
- Intermediate time
  - Complex behavior
  - Voltage relationship to ρ is changing
- Late time, $v(t) \propto \left(1/\rho^{3/2}\right)\left(1/t^{5/2}\right)$
  - Diffusion depth greater than T-R separation
  - Voltage is $\propto 1/t^{5/2}$
  - Voltage is $\propto 1/\rho^{3/2}$
  - Voltage is independent of r

Time-Domain Apparent Resistivity Values

In contrast to the incident EM plane wave response of a homogenous half-space which yields equations that can be easily solved for the true resistivity (therefore, defining a unique apparent resistivity, e.g., in magnetotellurics), the usual time-domain response equations do not allow a single "all-time" apparent resistivity solution. Therefore, it is customary to use asymptotic relations to define early-time (or stage) and late-time (or stage) formulations to calculate two apparent resistivities (see Table 2 above from Spies and Frishknecht, 1991).

Figure 19 from Fitterman and Labson (2005) plots such apparent resistivities for a central loop array over a $0.25\,\Omega\,\mathrm{m}$ half-space. Here, the corresponding apparent resistivity values approach the true resistivity at the earliest and latest times but they are far removed from it in the intermediate time range.

Late-time relations are usually used for deeper TEM soundings, e.g., for the Zonge ZeroTEM system, therefore, it is important to recognize unexpected features when using such resistivity soundings. Three such plots from Fitterman and Labson (2005) are shown for simple two-layer earth models in Figure 20. Besides showing unreasonably high apparent resistivities at early-times in all cases, there are unexpected "false" (misleading) undershoots, overshoots, and slope changes in the apparent resistivity values. Hence, it is easy to incorrectly infer basic layer sequences without rigorous inversions.

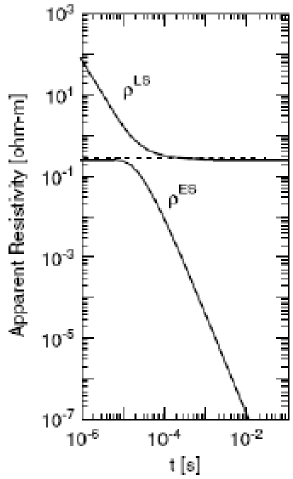


Figure 19. Central-loop (transmitter loop radius, $r = 50\,\mathrm{m}$) apparent resistivity curves for early- stage (ES) and late-stage (LS) sounding of a $0.25\,\Omega\,\mathrm{m}$, homogeneous half-space (from Fitterman and Labson, 2005).

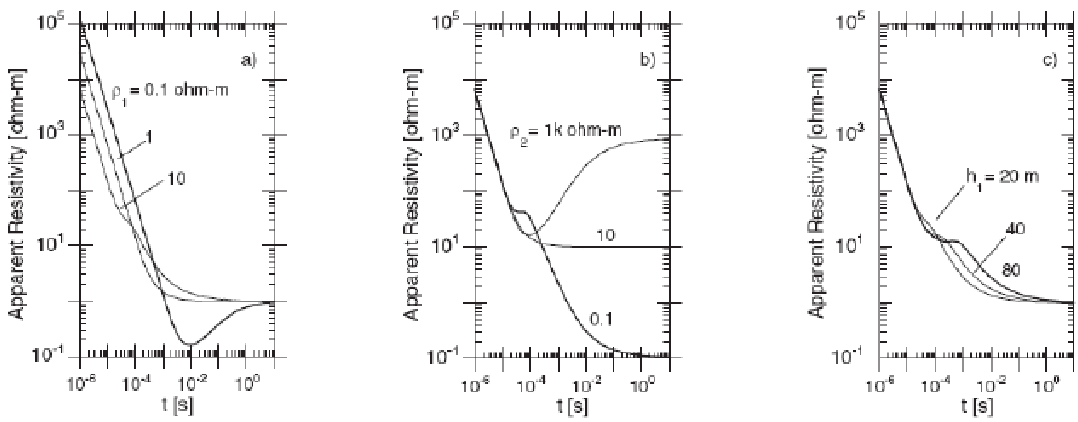
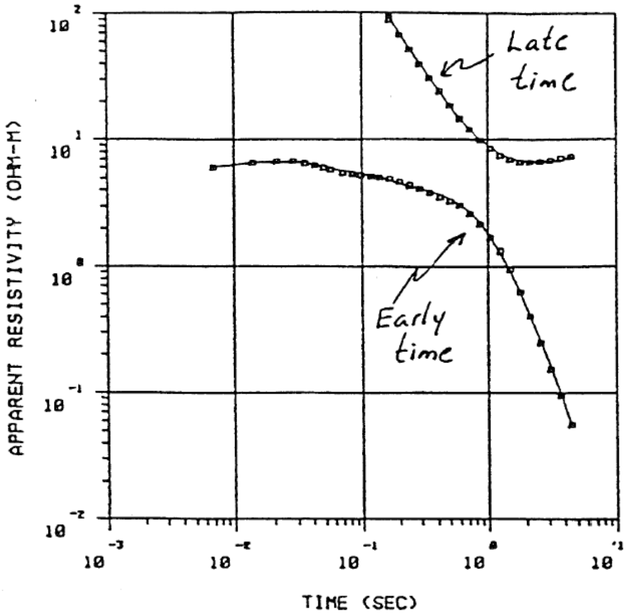


Figure 20. Central-loop, two-layer, late-stage apparent resistivity curves with $r = 50\,\mathrm{m}$ for a) variation of ρ_1 with $\rho_2 = 1\,\Omega\,\mathrm{m}$ and $h_1 = 20\,\mathrm{m}$, b) variation of ρ_2 with $\rho_1 = 10\,\Omega\,\mathrm{m}$ and $h_1 = 20\,\mathrm{m}$, and c) variation of h_1 with $\rho_1 = 10\,\Omega\,\mathrm{m}$ and $\rho_2 = 1\,\Omega\,\mathrm{m}$ (from Fitterman and Labson, 2005).

Sample Early and Late time ρ_a curves



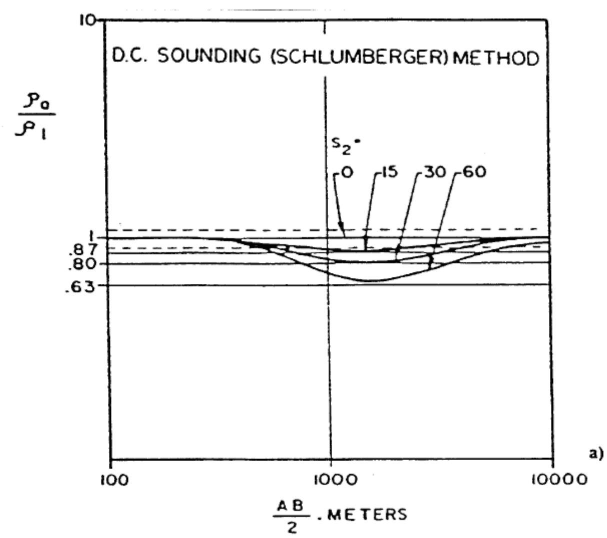
INTERPRETED MODEL:

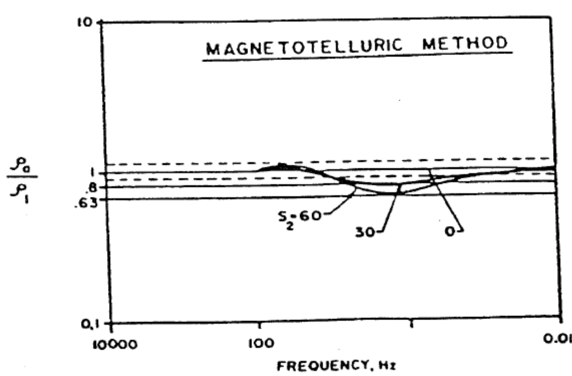
RESISTIVITY (OHM-M)	THICKNESS (M)
5.35	202.
71.8	146.
3.73	1739.
60.5	

TDEM sounding inversion. *Squares represent field data — resistivities as a function of time. Model is used to calculate resistivity curve that fits the field data with the smallest amount of error, e.g., 2.12%. Solid line is a curve calculated from the best fit model.*

Integrated GeoSciences Inc. (undated)

Comparison of DC, MT, and TEM ρ_a curves





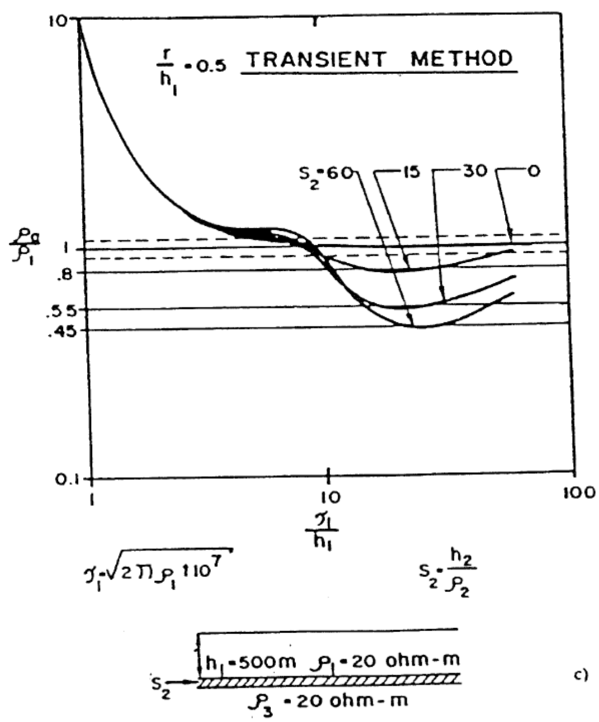


FIGURE 10. Computed apparent resistivity curves for geoelectric section sho[w]
a) direct current
b) magnetotelluric
c) transient

(Geo-physi-con, undated)

Response of Confined Target (Conducting Sphere)

Due to the "skin effect" the induced eddy currents will be initially ($t \sim 0$) confined to the surface of the sphere. Magnitude and direction of the eddy currents at $t \sim 0^+$ will be to maintain the uniform, original magnetic field in the sphere.

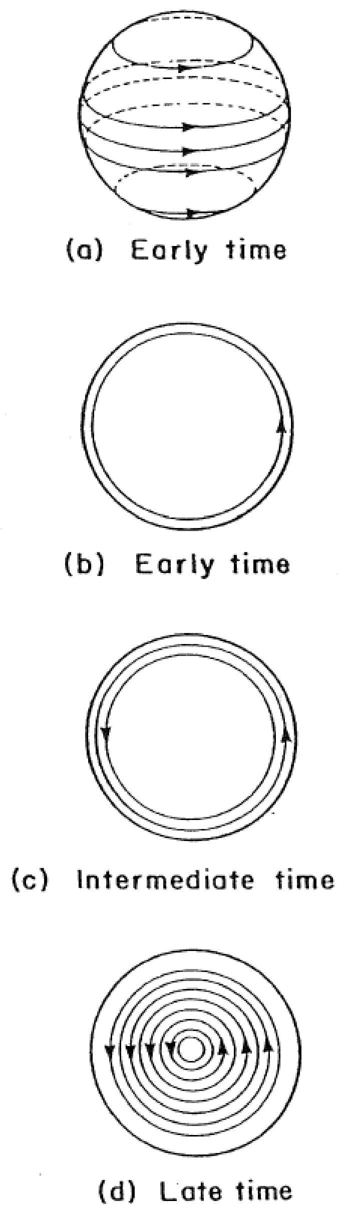


FIGURE 6. Sphere currents at various times (a – surface currents; b–d – equatorial/plane currents).

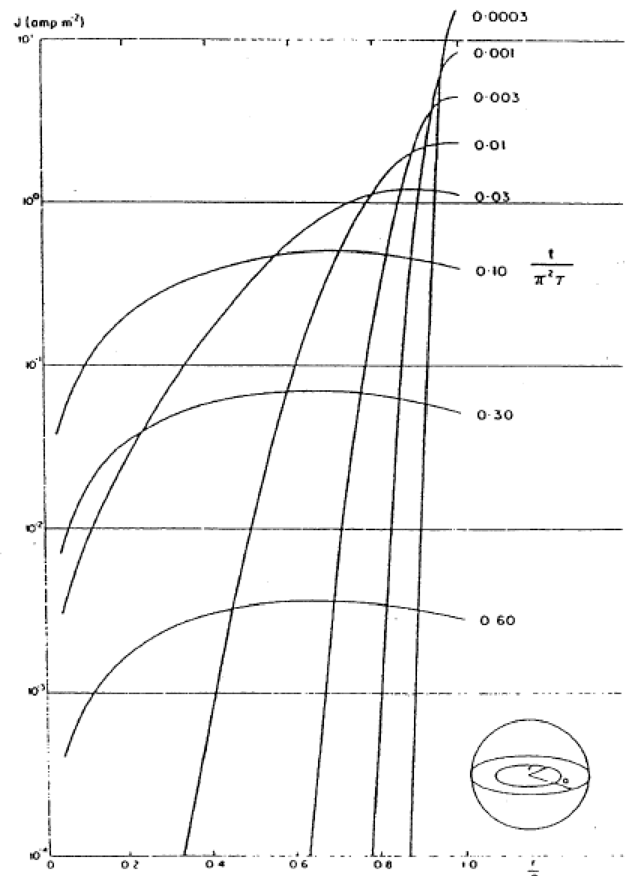


FIGURE 7. Radial distribution of sphere currents.

Mc Neill (1980)

Response of a Confined Target in a Conductive Media

(West and Macnae, 1991)

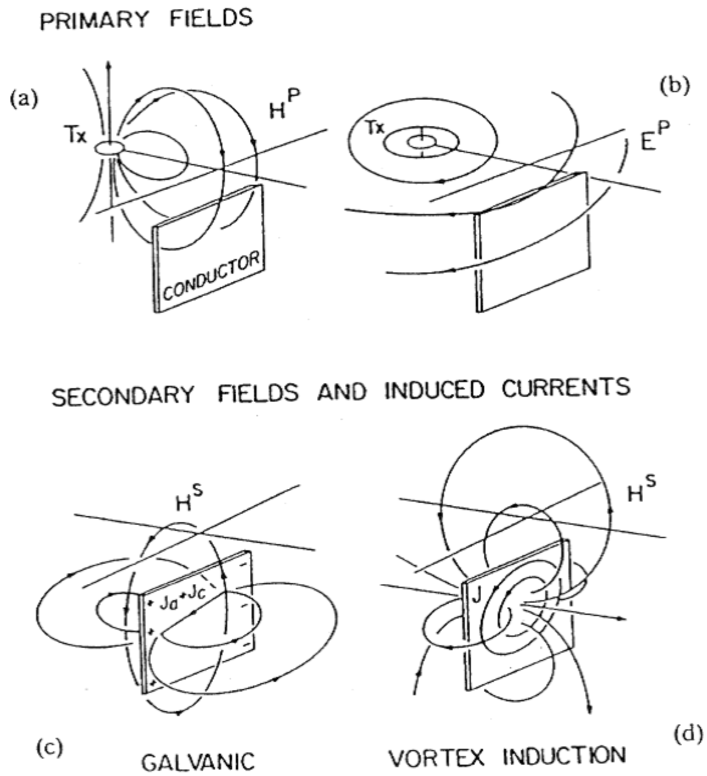


Fig. 30. Sketches of galvanic and vortex induction in a conductive zone. (a) primary magnetic field, (b) primary electric field, (c) induced galvanic current flow and its associated secondary magnetic field, (d) induced vortex current flow and its secondary magnetic field.

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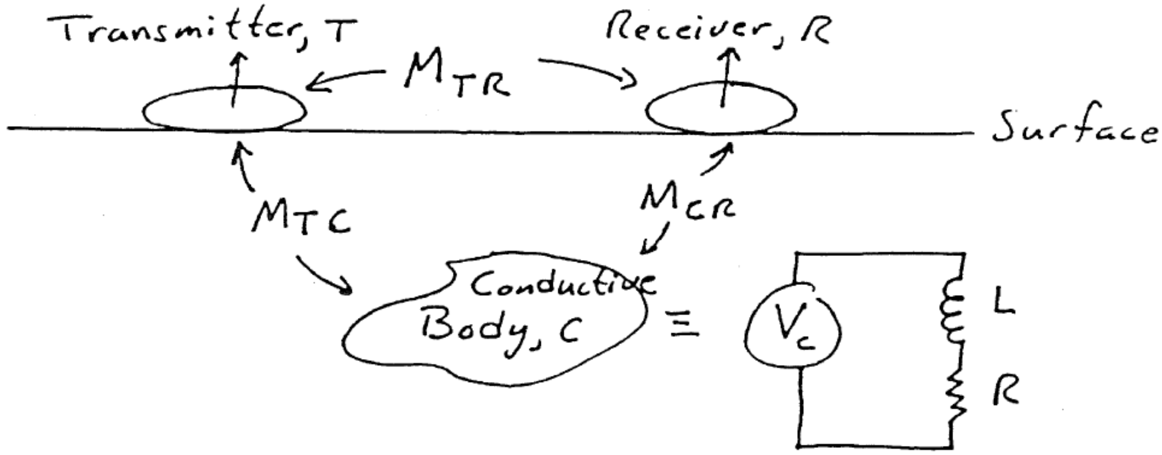
Appendix A

TEM Step Response of the Circuit Analogy

(after Grant and West, 1965, p. 541-543)



According to Kirchhoff's second law (voltage around a closed loop is zero)

$$V_C - V_L - V_R = 0$$

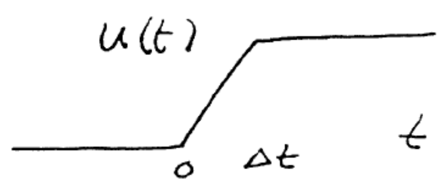
$$-M_{TC}\frac{d}{dt}I_T - \left[(R)(I_C) + L\frac{d}{dt}I_C\right] = 0.$$

Let I_T be a step current defined by

$$I_T = I_0 u(t)$$

where

$$\begin{aligned} u(t) &= 0,\ \ t < 0 \\ &= t/\Delta t,\ \ 0 \le t \le \Delta t \\ &= 1,\ \ t > \Delta t. \end{aligned}$$



Now

$$V_C(t) = -M_{TC}\frac{d}{dt}\left[(I_0)\,u(t)\right],$$

caused by the transmitter current, and

$$-V_L(t) - V_R(t) = -R\left[I_C(t)\right] - L\frac{d}{dt}\left[I_C(t)\right],$$

caused by the current in the conductor, C.

Kirchhoff's second law is now

$$-\left[L\frac{d}{dt}+R\right]I_C(t)=(M_{TC})\,(I_0)\,\frac{d}{dt}\left[u(t)\right].$$

1. Before switching on the current in T

$$I_C(t)=0,\; t<0.$$

2. During the interval $0\le t\le \Delta t$

$$-\left[L\frac{d}{dt}+R\right]I_C(t)=\frac{(M_{TC})\,(I_0)}{\Delta t}$$

which has the solution

$$I_C(t)=-\frac{(M_{TC})\,(I_0)}{(R)\,(\Delta t)}+Ae^{-\frac{R}{L}t}.$$

In order for $I_C(t)$ to be zero at $t=0$, the constant A must be

$$A=+\frac{(M_{TC})\,(I_0)}{(R)\,(\Delta t)}.$$

Therefore,

$$I_C(t)=-\frac{(M_{TC})\,(I_0)}{(R)\,(\Delta t)}\left[1-e^{-\frac{R}{L}t}\right]$$

At small values of t ($t\ll L/R$)

$$I_C(t)\simeq-\frac{(M_{TC})\,(I_0)}{(R)\,(\Delta t)}\frac{R}{L}t$$

after using

$$e^{-\frac{R}{L}t}=1-\frac{R}{L}t+\cdots.$$

In particular, at $t=\Delta t$

$$I_C(\Delta t)\simeq-\frac{(M_{TC})\,(I_0)}{L}.$$

3. When $t>\Delta t$,

$$u(t)=1$$

and

$$\frac{d}{dt}\left[u(t)\right]=0$$

so

$$\left[L\frac{d}{dt}+R\right]I_C(t)=0.$$

This has a solution

$$I_C(t)=Be^{-\frac{R}{L}t}.$$

The constant B is determined by applying the condition

$$I_C(\Delta t)\simeq-\frac{(M_{TC})\,(I_0)}{L}.$$

Since Δt is very small (near zero, $\Delta t\ll L/R$),

$$I_C(t)\simeq-\frac{(M_{TC})\,(I_0)}{L}\left[e^{-\frac{R}{L}t}\right]$$

when

$$t>\Delta t.$$

Voltages at R caused by T (primary) and C (secondary)

$$\begin{aligned} V_R^P(t) &= -M_{TR}\frac{d}{dt}\left[I_0 u(t)\right] \\ &= -\left(M_{TR}\right)\left(I_0\right)\left[\delta(t)\right] \end{aligned}$$

$$V_R^S(t) = -M_{CR}\frac{d}{dt}\left[I_C(t)\right]$$

where

$$I_C(t) \simeq -\underbrace{\frac{M_{TC}I_0}{\Delta t}\frac{t}{L}}_{0 \le t \le \Delta t} - \underbrace{\frac{M_{TC}I_0}{L}e^{-\frac{R}{L}t}}_{t > \Delta t}$$

Therefore,

$$V_R^S(t) = +\underbrace{\frac{\left(M_{TC}\right)\left(M_{CR}\right)}{L}I_0\delta(t)}_{0 \le t \le \Delta t} - \underbrace{\frac{\left(M_{TC}\right)\left(M_{CR}\right)}{L}I_0\left[\frac{R}{L}e^{-\frac{R}{L}t}\right]}_{t > \Delta t}.$$

Above,

$$\begin{aligned} \delta(t) &= 0,\ \ t < 0 \\ &= 1/\Delta t,\ \ 0 \le t \le \Delta t \\ &= 0,\ \ t > \Delta t \end{aligned}$$

Graphically